Unsubtracted Dispersion Relations in Weak Interactions and the Goldberger-Treiman Relation*

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A theory of weak interactions is formulated in terms of unsubtracted dispersion relations and is applied to the calculation of the π-μ decay amplitude. First, derivation of the Goldberger-Treiman relation is reexamined and the relationship between two different approaches, one by Goldberger and Treiman and the other by Gell-Mann and others, is studied based on Ida's formulation of this problem. Then, it is shown that a consistent use of unsubtracted dispersion relations for weak interactions leads us to an eigenvalue restriction to be imposed on the choice of the parameters in strong interactions.

I. INTRODUCTION

Recently, a dynamical approach to weak interactions was proposed by McCliment and Nishijima[1] and was applied to derivation of the selection rule $|\Delta I|=\frac{1}{2}$ in nonleptonic decays of strange particles. In the present paper the Goldberger-Treiman relation in $\pi-\mu$ decay is derived along the line of approach employed in Ref. 1. First we shall recapitulate the basic ideas underlying this approach and then point out some puzzling points in the derivation of the Goldberger-Treiman relation.

In field theory, a distinction is usually made between elementary and composite particles although rigorous definitions of them are not yet known, and a similar distinction is made between fundamental and induced interactions. In Lagrangian theory both elementary fields and fundamental interactions are defined as those objects occurring in the original Lagrangian, and those others not occurring in it are called composite fields and induced interactions, respectively.

In dispersion theory, fundamental and induced interactions are characterized by the presence or the absence of subtractions in the dispersion relations for the corresponding vertex functions, and a composite particle is likewise characterized by all the vertices involving it—except for the universal electromagnetic interactions—being induced ones.

In quantum electrodyanmics, for instance, the Dirac-type interactions are considered to be fundamental, whereas the Pauli-type interactions are treated as induced ones, so that once-subtracted dispersion relations are used for the former or the charge form factors and unsubtracted dispersion relations are assumed for the latter or the magnetic form factors. These conditions represent the consequences of the so-called principle of minimal electromagnetic interactions.

In the next step these classifications of particles and of interactions are combined with the requirement of renormalizability, i.e., it is postulated that the Lagrangian describing strong interactions is renormalizable in the conventional sense. If we take this for granted, the deuteron is not an elementary particle since the interactions of a particle with unit spin are not renormalizable. This also leads to an interesting conclusion that all the weak interactions are induced ones since, as far as we know, the properties of weak interactions cannot be described in terms of renormalizable fundamental interactions. From the outset of classifications of particles and of interactions some similarities are expected to exist between composite particles and induced interactions, and this anticipation is just the motivation of the present work. In what follows we shall list the characteristic features of the present dynamical approach.

(1) The deuteron has unit spin, but it can be accommodated in the renormalizable theory of strong interactions provided that its interactions with other fields are induced ones, since only fundamental unrenormalizable interactions are supposed to give rise to unmanageable divergences. The same remark applies to weak interactions, and even when they could not be represented by fundamental renormalizable interactions they would not give us divergence difficulties provided all weak interactions are induced ones.

(2) Since the deuteron is a bound state it can exist only when certain eigenvalue restrictions are satisfied, this must also be the case for weak interactions. The Lagrangian representing strong interactions possesses various symmetry properties, e.g., space-reflection invariance, charge-conjugation invariance, and conservation of strangeness, and there are no terms in the original Lagrangian that correspond to induced weak interactions and violate those symmetries. Under normal conditions weak interactions would not be induced and the only possibility for induced weak interactions would be a self-consistent bootstrap mechanism. The self-consistency conditions are expressed in the form of eigenvalue equations to determine fundamental parameters in strong interactions.

(3) The deuteron exists only in the ${}^3S_1+{}^3D_1$ state but not in other states since the eigenvalue restriction is satisfied only in this state. A similar situation is expected to persist for weak interactions. They can be induced only in those states in which eigenvalue restric-

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[1] E. R. McCliment and K. Nishijima, Phys. Rev. 128, 1970 (1962).

tions are satisfied and this property may be expressed in the form of selection rules in weak interactions.

In Ref. 1 these points have been investigated in detail for nonleptonic decays of hyperons. It was concluded that the well-known selection rule $|\Delta I|=\frac{1}{2}$ is a consequence of charge independence of strong interactions, and it was also shown that the interactions obeying this selection rule can be induced in a self-consistent manner only when fundamental parameters in strong interactions are properly chosen, i.e.,

$$G_{\Sigma\Lambda\pi}^2/4\pi\approx 7\,,\quad G_{\Sigma\Sigma\pi}^2/4\pi\approx 1\,. \tag{1.1}$$

The results above illustrate the essence of the present approach.

Now our subject will be switched to leptonic decays of nonstrangle particles. The simplest process of this kind is $\pi-\mu$ decay, and, in fact, investigation of this process done by Goldberger and Treiman[2,3] was the first step toward understanding of weak interactions in terms of dispersion relations. They derived a relation, called after their names, connecting the $\pi-\mu$ decay constant with the Fermi coupling constant. The excellent agreement of this formula with experiment stimulated many other investigations regarding its derivation. In their original derivation Goldberger and Treiman assumed an unsubtracted dispersion relation for the decay amplitude of the charged pion, whereas Gell-Mann and others[4,5] assumed an unsubtracted dispersion relation for the pseudoscalar amplitude in nuclear capture of the μ meson and obtained the same result. These two assumptions are not necessarily equivalent however, and one of the main objects of this paper is to clarify the reason these two inequivalent assumptions lead to the same goal.

In Sec. II dispersion relations obeyed by weak pseudoscalar amplitudes are written down, and in Sec. III the dispersion relation for the $\pi-\mu$ decay amplitude is given and the Goldberger-Treiman relation is derived in the multichannel case. It has already been pointed out by Barrett and Barton[6] and by Ida[7] that the solution of this problem obtained by Goldberger and Treiman in the nucleon-antinucleon pair approximation does not satisfy the originally assumed unsubtracted dispersion relation when the wave-function renormalization constant of the pion field is divergent. In order to overcome this difficulty, this problem is formulated in the multichannel case in Sec. IV, and an approximate solution is sought in Sec. V. Then, in Sec. VI the approximate solution is inserted into Ida's formula and an eigenvalue restriction is obtained as we originally expected. It is worthwhile to notice, however, that a consistent use of unsubtracted dispersion relations does not necessarily give eigenvalue restrictions, but the possibility of obtaining eigenvalue restrictions depends critically on the type of coupling. A discussion of this point is made in Sec. VII.

II. DISPERSION RELATIONS FOR PSEUDOSCALAR AMPLITUDES

The interaction Hamiltonian density for β decay or μ capture is usually given by

$$\begin{aligned}H=&(C_V\bar\psi_p\gamma_\lambda\psi_n-C_A\bar\psi_p\gamma_\lambda\gamma_5\psi_n)\bar\psi_l\gamma_\lambda(1+\gamma_5)\psi_\nu\\&+(C_V\bar\psi_n\gamma_\lambda\psi_p-C_A\bar\psi_n\gamma_\lambda\gamma_5\psi_p)\bar\psi_\nu\gamma_\lambda(1+\gamma_5)\psi_l\,,\end{aligned} \tag{2.1}$$

where ψ_l stands for the lepton field. In this paper only the axial-vector part will be investigated, and that part of the Hamiltonian density will be denoted by

$$H_A=-A_\lambda\bar\psi_\nu i\gamma_\lambda(1+\gamma_5)\psi_l-A_\lambda^\dagger\bar\psi_l i\gamma_\lambda(1+\gamma_5)\psi_\nu\,. \tag{2.2}$$

If A_λ should be expressed in terms of field operators, we should write down an expression like (2.1), but it is not necessary and might eventually be impossible to do so. Instead of doing so we shall determine the matrix elements of A_λ from other general requirements: (1) $A_\lambda(x)$ is a local axial-vector field, and (2) all the matrix elements of $A_\lambda(0)$ between the vacuum and other states regarded as functions of the total barycentric energy squared s satisfy unsubtracted dispersion relations.

Rigorously speaking the second condition is not valid for many-particle states, but practically it is valid since only two-particle states are considered in this paper. This incompleteness is a reflection of the fact that dispersion theory as it stands is not yet a closed theory. For a rigorous formulation of the second condition, reference should be made to parametric dispersion relations for Green's functions, but for an approximate treatment of the problem ordinary dispersion relations are more practical.

When the two conditions above are satisfied, (2.2) represents an effective Hamiltonian for the induced interaction rather than the actual Hamiltonian for a fundamental interaction.

The normal decay rate of the charged pion is determined by the matrix element

$$\langle\pi^-|A_\lambda(0)|0\rangle=\frac{1}{(2q_0)^{1/2}}iq_\lambda F\,, \tag{2.3}$$

where q is the four-momentum of the π^- meson. The decay rate for the process

$$\pi\to\mu+\nu \tag{2.4}$$

is given by

$$\frac{1}{\tau}=m_\pi\left(1-\frac{m_\mu^2}{m_\pi^2}\right)^2\frac{(m_\mu F)^2}{4\pi}\,. \tag{2.5}$$

[2] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958).
[3] M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958).
[4] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962). See also other papers quoted there.
[5] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960).
[6] B. Barrett and G. Barton, Nuovo Cimento **29**, 703 (1963).
[7] M. Ida, Phys. Rev. **132**, 401 (1963).

Ida[7] suggested the use of $\partial_\lambda A_\lambda$ as the fundamental object in the present problem and we shall follow his suggestion by defining

$$\Phi=\partial_\lambda A_\lambda. \tag{2.6}$$

Its matrix element for pion decay is given by

$$\langle\pi^-|\Phi(0)|0\rangle=\frac{1}{(2q_0)^{1/2}}q^2F=-\frac{1}{(2q_0)^{1/2}}\mu^2F, \tag{2.7}$$

where μ denotes the pion rest mass hereafter.

Next, a nucleon form factor associated with this pseudoscalar field is defined by

$$\langle n\bar{p},(-)|\Phi(0)|0\rangle=\bar{u}(n)i\gamma_5 v(\bar{p})f_{n\bar{p}}(s), \tag{2.8}$$

where $s=-(n+\bar{p})^2$ is the total barycentric energy squared of the $n\bar{p}$ system, and $u(n)$ and $v(\bar{p})$ are the Dirac spinors for the neutron and the antiproton in the final state. In fixing the phase of the state $|n\bar{p}(-)\rangle$ we use the convention

$$|n\bar{p},(-)\rangle=a^\dagger(n)^{\text{out}}a^\dagger(\bar{p})^{\text{out}}|0\rangle.$$

This form factor is normalized at $s=0$ by

$$f_{n\bar{p}}(0)=2Mg_A, \tag{2.9}$$

where M denotes the nucleon rest mass, and g_A is the renormalized axial-vector coupling constant in β decay and μ capture, and is related to C_A in (2.1) by

$$g_A=-C_A. \tag{2.10}$$

The absorptive part of the form factor obeys the unitarity condition

$$\text{Abs}\langle\alpha,(-)|\Phi(0)|0\rangle$$
$$=-\frac{(2\pi)^4}{2}\sum_\beta T_{\alpha\beta}^\dagger\delta^4(P_\alpha-P_\beta)\langle\beta,(-)|\Phi(0)|0\rangle, \tag{2.11}$$

where T is related to the S matrix by

$$S_{\alpha\beta}=\delta_{\alpha\beta}-i(2\pi)^4\delta^4(P_\alpha-P_\beta)T_{\alpha\beta}. \tag{2.12}$$

Since Φ is a pseudoscalar, the least massive intermediate state in (2.11) is the one-pion state. Choosing $\alpha=n\bar{p}$, and $\beta=\pi^-$, the contribution of the one-pion intermediate state to the absorptive part of $\langle\alpha,(-)|\Phi(0)|0\rangle$ is evaluated:

$$\text{Abs}\langle\alpha,(-)|\Phi(0)|0\rangle$$
$$=-\frac{(2\pi)^4}{2}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2q_0}\bar{u}(n)iG\sqrt{2}\gamma_5 v(\bar{p})$$
$$\times(-\mu^2)F\delta^4(n+\bar{p}-q), \tag{2.13}$$

where use has been made of the relations

$$T_{n\bar{p},\pi^-}=\bar{u}(n)iG\sqrt{2}\gamma_5 v(\bar{p})\frac{1}{(2q_0)^{1/2}},$$

$$\langle\pi|\Phi(0)|0\rangle=\frac{1}{(2q_0)^{1/2}}(-\mu^2)F.$$

Thus, the contribution of the one-pion intermediate state to the absorptive part of $f_{n\bar{p}}(s)$ is given by

$$\text{Im}f_{n\bar{p}}(s)=\pi\sqrt{2}GF\mu^2\delta(s-\mu^2). \tag{2.14}$$

Therefore, the dispersion relation for $f_{n\bar{p}}(s)$ is given by

$$f_{n\bar{p}}(s)=\frac{\sqrt{2}GF\mu^2}{\mu^2-s}+\frac{1}{\pi}\int_{(3\mu)^2}^\infty ds'\frac{\text{Im}f_{n\bar{p}}(s')}{s'-s-i\epsilon}. \tag{2.15}$$

In order to take account of the boundary condition (2.9), the above dispersion relation will be written in the form of a once-subtracted dispersion relation.

$$f_{n\bar{p}}(s)=2Mg_A+\frac{\sqrt{2}GFs}{\mu^2-s}+\frac{s}{\pi}\int_{(3\mu)^2}^\infty ds'\frac{\text{Im}f_{n\bar{p}}(s')}{s'(s'-s-i\epsilon)}. \tag{2.16}$$

The above dispersion relation for the $n\bar{p}$ channel can be generalized to an arbitrary channel α.

$$f_\alpha(s)=f_\alpha(0)+\frac{G_\alpha Fs}{\mu^2-s}+\frac{s}{\pi}\int_{(3\mu)^2}^\infty ds'\frac{\text{Im}f_\alpha(s')}{s'(s'-s-i\epsilon)}, \tag{2.17}$$

where $f_\alpha(s)$ denotes the form factor of Φ in the channel α, e.g.,

$$\langle\alpha,(-)|\Phi(0)|0\rangle=c_\alpha f_\alpha(s),\quad T_{\alpha,\pi^-}=c_\alpha G_\alpha,$$

where c_α is an invariant in the channel α, and G_α is the coupling constant between the π^- meson and the channel α, e.g.,

$$c_{n\bar{p}}=\bar{u}(n)i\gamma_5 v(\bar{p}),\quad G_{n\bar{p}}=\sqrt{2}G.$$

The dispersion relation (2.16) is reminiscent of that for $\square\varphi$, i.e., the form factor $h_{n\bar{p}}(s)$ defined by

$$\langle n\bar{p},(-)|\square\varphi|0\rangle=\bar{u}(n)i\gamma_5 v(\bar{p})h_{n\bar{p}}(s) \tag{2.18}$$

satisfies a dispersion relation

$$h_{n\bar{p}}(s)=-\frac{\sqrt{2}Gs}{\mu^2-s}+\frac{s}{\pi}\int_{(3\mu)^2}^\infty ds'\frac{\text{Im}h_{n\bar{p}}(s')}{s'(s'-s-i\epsilon)}. \tag{2.19}$$

By generalizing this dispersion relation to an arbitrary channel α we may write

$$h_\alpha(s)=-\frac{G_\alpha s}{\mu^2-s}+\frac{s}{\pi}\int_{(3\mu)^2}^\infty ds'\frac{\text{Im}h_\alpha(s')}{s'(s'-s-i\epsilon)}. \tag{2.20}$$

Let us consider the combination

$$L_\alpha(s)=f_\alpha(s)+Fh_\alpha(s), \tag{2.21}$$

then the new functions $L_\alpha(s)$ satisfy dispersion relations of the form

$$L_\alpha(s)=f_\alpha(0)+\frac{s}{\pi}\int_{(3\mu)^2}^{\infty} ds'\frac{\mathrm{Im}L_\alpha(s')}{s'(s'-s-i\epsilon)}. \quad (2.22)$$

It is worthwhile to mention that the pole term corresponding to the one-pion intermediate state is absent in the above dispersion relation. It is also the case for the form factor associated with the pion current $j_\pi=(\Box-\mu^2)\varphi$. The pion vertex functions $K_\alpha(s)$ are defined by

$$\langle\alpha,(-)|j_\pi(0)|0\rangle=c_\alpha K_\alpha(s). \quad (2.23)$$

Then it is clear that $K_\alpha(s)$ is related to $h_\alpha(s)$ by

$$K_\alpha(s)=[(s-\mu^2)/s]h_\alpha(s). \quad (2.24)$$

The form factors $L_\alpha(s)$ are the matrix elements of the operator

$$\Phi+F\Box\varphi=\partial_\lambda(A_\lambda+F\partial_\lambda\varphi), \quad (2.25)$$

and can be expressed by

$$L_\alpha(s)=f_\alpha(s)+F[s/(s-\mu^2)]K_\alpha(s). \quad (2.26)$$

Owing to the absence of the pion-pole term in dispersion relations, it is simpler to study $L_\alpha(s)$ than $f_\alpha(s)$, so that we shall regard the form factors $L_\alpha(s)$ as more basic than the original $f_\alpha(s)$.

Subtractions

In determining the form factors $L_\alpha(s)$ it is essentially important to fix the number of subtractions needed in each dispersion relation. For this purpose we shall utilize the similarity between the form factors L_α and K_α and the following postulate:

Postulate I. Form factors corresponding to renormalizable vertices satisfy once-subtracted dispersion relations, and all others satisfy unsubtracted dispersion relations.

The argument leading to this postulate has been given in Ref. 1 so that we shall not repeat it, but we shall mention that the subtractions in dispersion relations correspond exactly to those needed in the vertex renormalization. Postulate I can immediately be applied to form factors K_α and then to L_α by making use of the similarity between them.

First, the unitarity conditions for K_α and L_α will be written down to visualize their similarity.

$$c_\alpha\,\mathrm{Im}K_\alpha(s)=-\frac{(2\pi)^4}{2}\sum_\beta{}' T_{\alpha\beta}{}^\dagger\delta^4(P_\alpha-P_\beta)c_\beta K_\beta(s), \quad (2.27)$$

$$c_\alpha\,\mathrm{Im}L_\alpha(s)=-\frac{(2\pi)^4}{2}\sum_\beta{}' T_{\alpha\beta}{}^\dagger\delta^4(P_\alpha-P_\beta)c_\beta L_\beta(s), \quad (2.28)$$

where the primes on the summation symbols mean omission of the one-pion intermediate state in the summation. Therefore, they satisfy unitarity conditions of the same form.

From postulate I we know that subtractions are needed for K_α for $\alpha=\bar{B}B$ (baryon-antibaryon pair), 3π, and $K\bar{K}\pi$. Thus the dispersion relations for K_α are given by

$$K_\alpha(s)=G_\alpha+\frac{s-\mu^2}{\pi}\int ds'\frac{\mathrm{Im}K_\alpha(s')}{(s'-\mu^2)(s'-s-i\epsilon)},$$
$$\text{for}\quad \alpha=\bar{B}B,\,3\pi,\,\bar{K}K\pi, \quad (2.29)$$

and

$$K_\alpha(s)=\frac{1}{\pi}\int ds'\frac{\mathrm{Im}K_\alpha(s')}{s'-s-i\epsilon},\quad \text{for all other channels.} \quad (2.30)$$

Therefore, we shall assume that the form factor $L_\alpha(s)$ requires the same number of subtractions as does $K_\alpha(s)$ for the same channel α.

$$L_\alpha(s)=f_\alpha(0)+\frac{s}{\pi}\int ds'\frac{\mathrm{Im}L_\alpha(s')}{s'(s'-s-i\epsilon)},$$
$$\text{for}\quad \alpha=\bar{B}B,\,3\pi,\,\bar{K}K\pi, \quad (2.31)$$

and

$$L_\alpha(s)=\frac{1}{\pi}\int ds'\frac{\mathrm{Im}L_\alpha(s')}{s'-s-i\epsilon},\quad \text{for all other channels.} \quad (2.32)$$

These dispersion relations form the basis of a dynamical calculation of form factors to be studied in later sections.

III. DISPERSION RELATION FOR THE DECAY AMPLITUDE

Combination of the unitary condition (2.28) and dispersion relations (2.31) and (2.32) enables us to determine, if not uniquely, the matrix elements of Φ when the subtraction constants $f_\alpha(0)$ and the decay constant F are given. The decay constant F is defined by (2.7), i.e.,

$$\langle\pi^-|\Phi(0)|0\rangle=\langle0|\Phi(0)|\pi^+\rangle=-[\mu^2/(2q_0)^{1/2}]F. \quad (2.7)$$

This formula can be extended to the off-shell pion "π" of mass $(s)^{1/2}$ as given by

$$\langle\text{“}\pi^-\text{”},(-)|\Phi(0)|0\rangle=-[s/(2q_0)^{1/2}]F(s). \quad (3.1)$$

A rigorous definition of $F(s)$ can be given, with the help of the LSZ reduction formula,[8] by

$$sF(s)=i\int d^4z\, e^{-iqz}(\Box_z-\mu^2)\langle0|T[\Phi(0)\varphi^\dagger(z)]|0\rangle, \quad (3.2)$$

where $s=-q^2$.

The unitarity condition (2.11) as applied to $\alpha=$"π^-" gives

$$s\,\mathrm{Im}F(s)=\frac{(2\pi)^4}{2}\sum_\alpha{}' K_\alpha{}^*(s)f_\alpha(s)c_\alpha{}^*c_\alpha\delta^4(q-P_\alpha). \quad (3.3)$$

[8] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento 1, 205 (1955).

Let us introduce a density function for the channel α by

$$\rho_\alpha(s)=(2\pi)^3 \sum_{\text{in}\alpha}{}' c_\alpha{}^* c_\alpha \delta^4(q-P_\alpha)\,; \tag{3.4}$$

then Eq. (3.3) may be written in a simpler form

$$s\,\text{Im}F(s)=\pi \sum_\alpha{}' K_\alpha{}^*(s) f_\alpha(s)\rho_\alpha(s)\,. \tag{3.5}$$

This function $F(s)$ is related to the constant F by

$$F(s)=F+\frac{s-\mu^2}{\pi}\int_{(3\mu)^2}^\infty ds' \frac{\text{Im}F(s')}{(s'-\mu^2)(s'-s-i\epsilon)}\,, \tag{3.6}$$

independently of whether or not $F(s)$ requires a subtraction.

In what follows $F(s)$ is assumed to satisfy an unsubtracted dispersion relation.

$$F(s)=\frac{1}{\pi}\int_{(3\mu)^2}^\infty ds' \frac{\text{Im}F(s')}{s'-s-i\epsilon}\,. \tag{3.7}$$

Then F can be expressed in terms of other parameters as done by Goldberger and Treiman.[2] Here we shall closely follow Ida's derivation[7] which is a generalization of that of Goldberger and Treiman. For this purpose we recall the formula

$$(s-\mu^2)^2\sigma(s)=\sum_\alpha{}' K_\alpha{}^*(s) K_\alpha(s)\rho_\alpha(s)\,, \tag{3.8}$$

where $\sigma(s)$ is the Lehmann weight function for the pion propagator. A similar function $\gamma(s)$ will be introduced by

$$(s-\mu^2)^2\gamma(s)=\sum_\alpha{}' K_\alpha{}^*(s) L_\alpha(s)\rho_\alpha(s)\,. \tag{3.9}$$

Then it is possible, with the help of the relation (2.26), to express $\text{Im}F(s)$ in terms of these functions, i.e.,

$$s\,\text{Im}F(s)=\pi(s-\mu^2)^2\left[\gamma(s)-F\frac{s}{s-\mu^2}\sigma(s)\right], \tag{3.10}$$

or

$$\frac{1}{\pi}\frac{\text{Im}F(s)}{s-\mu^2}=\frac{s-\mu^2}{s}\gamma(s)-F\sigma(s)\,. \tag{3.11}$$

The unsubtracted dispersion relation (3.7) enables us to write down the following relation:

$$F=F(\mu^2)=\int ds\frac{s-\mu^2}{s}\gamma(s)-F\int ds\sigma(s)\,. \tag{3.12}$$

Solving this equation for F, we find

$$F=\int ds\frac{s-\mu^2}{s}\gamma(s)\Big/\left[1+\int ds\sigma(s)\right]. \tag{3.13}$$

This is the generalized Goldberger-Treiman relation for the multichannel case first derived by Ida.[7] The denominator is equal to the wave-function renormalization constant of the pion field, i.e.,

$$Z_\pi{}^{-1}=1+\int ds\sigma(s)\,, \tag{3.14}$$

and it is considered to be divergent in most cases. When it is divergent, however, the relation (3.13) is meaningless and an alternative form of the solution of (3.12) is given by

$$F=\int ds\frac{s-\mu^2}{s}[\gamma(s)-F\sigma(s)]\Big/\left[1+\int ds\frac{\mu^2}{s}\sigma(s)\right]. \tag{3.15}$$

This formula was first derived by Ida[7] based on the N/D method, but in this paper it has been derived without reference to that method in order to clarify the subtraction properties of the dispersion relations. From now on throughout this paper we shall assume the divergence of $Z_\pi{}^{-1}$ so that we shall not use Eq. (3.13) but only the alternative form (3.15) will be used. The latter will be referred to as Ida's formula.

Next the properties of Ida's formula will be studied. First, the denominator of (3.15) is convergent provided the Lehmann representation without subtraction is valid. Therefore, the numerator must converge, too, if the formula (3.15) should be meaningful. At this point one can immediately recognize that the two terms in the numerator lead to divergent results if integrated separately, since the second term gives rise to the same divergence as that in $Z_\pi{}^{-1}$. This implies that the two terms in the integrand cancel one another for large values of s, i.e.,

$$\lim_{s\to\infty}\frac{\gamma(s)-F\sigma(s)}{\sigma(s)}=0\,, \tag{3.16}$$

or

$$F=\lim_{s\to\infty}\frac{\gamma(s)}{\sigma(s)}\,. \tag{3.17}$$

This relation expresses the convergence condition for Ida's formula provided the difference $\gamma(s)-F\sigma(s)$ does not oscillate infinite times for large values of s. The fact that the convergence condition (3.17) is nothing but the Goldberger-Treiman relation was first recognized by Barrett and Barton[6] in the nucleon-antinucleon approximation, and then by Ida in the general case. In what follows it will be shown that the Goldberger-Treiman relation follows from the convergence condition in typical models.

Nucleon-Antinucleon Pair Approximation

If only the nucleon-antinucleon pair is kept in the unitarity conditions (2.27) and (2.28), $K_{n\bar{p}}(s)$ and $L_{n\bar{p}}(s)$ satisfy equations of the same form.

Unitary: $\mathrm{Im}K_{n\bar{p}}(s)=[\tan\delta(s)]\,\mathrm{Re}K_{n\bar{p}}(s)$, (3.18)

$$\mathrm{Im}L_{n\bar{p}}(s)=[\tan\delta(s)]\,\mathrm{Re}L_{n\bar{p}}(s)\,, \quad (3.19)$$

where $\tan\delta$ is the ratio of the imaginary part to the real part of the amplitude for elastic $n\bar{p}$ scattering in the $^1S_0$ state.

Dispersion relations:

$$K_{n\bar{p}}(s)=\sqrt{2}G+\frac{s-\mu^2}{\pi}\int_{4M^2}^{\infty}ds'\frac{\mathrm{Im}K_{n\bar{p}}(s')}{(s'-\mu^2)(s'-s-i\epsilon)}\,, \quad (3.20)$$

$$L_{n\bar{p}}(s)=2Mg_A+\frac{s}{\pi}\int_{4M^2}^{\infty}ds'\frac{\mathrm{Im}L_{n\bar{p}}(s')}{s'(s'-s-i\epsilon)}\,. \quad (3.21)$$

Combining unitarity with dispersion relations equations of the Omnès type are obtained for both $K_{n\bar{p}}(s)$ and $L_{n\bar{p}}(s)$. If the simplest Omnès solutions are employed for both form factors, it is immediately concluded that these two functions differ only by a constant factor, i.e.,

$$\frac{L_{n\bar{p}}(s)}{K_{n\bar{p}}(s)}=\frac{L_{n\bar{p}}(0)}{K_{n\bar{p}}(0)}\approx\frac{L_{n\bar{p}}(0)}{K_{n\bar{p}}(\mu^2)}=\frac{2Mg_A}{\sqrt{2}G}\,. \quad (3.22)$$

In this approximation $\sigma(s)$ and $\gamma(s)$ are given by

$$(s-\mu^2)^2\sigma(s)=K_{n\bar{p}}^*(s)K_{n\bar{p}}(s)\rho_{n\bar{p}}(s)\,,$$
$$(s-\mu^2)^2\gamma(s)=K_{n\bar{p}}^*(s)L_{n\bar{p}}(s)\rho_{n\bar{p}}(s)\,,$$

so that application of the convergence condition (3.17) yields

$$F=\frac{\gamma(s)}{\sigma(s)}=\frac{L_{n\bar{p}}(0)}{K_{n\bar{p}}(0)}\approx\frac{2Mg_A}{\sqrt{2}G}\,. \quad (3.23)$$

This is the Goldberger-Treiman relation and the derivation above parallels that of Goldberger-Treiman[2,3] except that the convergence condition (3.17) was used here instead of (3.13). It should be noticed, however, that when the integrals in (3.13) are divergent or very slowly convergent the ratio tends to (3.17).

The Model of Gell-Mann and Others

Gell-Mann and others[9–11] proposed a model in which Φ is proportional to the pion field φ. The proportionality constant can be determined with reference to (2.7), i.e.,

$$\Phi=-\mu^2F\varphi\,. \quad (3.24)$$

From this equation we immediately find

$$f_\alpha(s)=-\mu^2F[K_\alpha(s)/(s-\mu^2)]\,, \quad (3.25)$$

and

$$L_\alpha(s)=FK_\alpha(s)\,. \quad (3.26)$$

The Goldberger-Treiman relation (3.23) is an immediate consequence of (3.26). It is worthwhile to notice, however, that in this model the unsubtracted dispersion relation for $F(s)$ or Ida's formula has not been used to derive the Goldberger-Treiman relation. We may conclude from the above illustrations that what is relevant in deriving the Goldberger-Treiman relation is the unsubtracted dispersion relations for the form factors $f_\alpha(s)$, but not necessarily the assumed unsubtracted dispersion relation for $F(s)$. In the nucleon-antinucleon pair approximation considered by Goldberger and Treiman, the unsubtracted dispersion relation for $f_{n\bar{p}}(s)$ is a consequence of that for $F(s)$ as seen below. In this approximation we have

$$\frac{\gamma(s)-F\sigma(s)}{\sigma(s)}=\frac{L_{n\bar{p}}(s)-FK_{n\bar{p}}(s)}{K_{n\bar{p}}(s)}\,, \quad (3.27)$$

and the convergence condition (3.17) requires

$$\lim_{s\to\infty}\frac{L_{n\bar{p}}(s)-FK_{n\bar{p}}(s)}{K_{n\bar{p}}(s)}=0\,, \quad (3.28)$$

or

$$\lim_{s\to\infty}\frac{f_{n\bar{p}}(s)}{K_{n\bar{p}}(s)}=0\,. \quad (3.29)$$

From the argument given in the first model it is easily seen that the ratio (3.28), before the limit is taken, is a constant and hence follows the relation

$$L_{n\bar{p}}(s)=FK_{n\bar{p}}(s)\,, \quad (3.30)$$

or

$$f_{n\bar{p}}(s)=-\mu^2F[K_{n\bar{p}}(s)/(s-\mu^2)]\,. \quad (3.31)$$

Therefore, when $K_{n\bar{p}}(s)$ requires only one subtraction $f_{n\bar{p}}(s)$ satisfies an unsubtracted dispersion relation as postulated by other people[4,5]; in other words, in this approximation the assumption of an unsubtracted dispersion relation for $F(s)$ made by Goldberger and Treiman already requires an unsubtracted dispersion relation for $f_{n\bar{p}}(s)$ assumed by others. The Goldberger-Treiman relation is essentially a consequence of the latter assumption but does not necessarily require the stronger assumption made by Goldberger and Treiman. In order to clarify this point let us insert the approximate formula (3.30) into Ida's formula (3.15). Then what we get is $F=0$, which is an indication that $F(s)$ requires a subtraction in contradiction to the original assumption. This difficulty has already been pointed out in Refs. 1 and 6, and the necessity of a subtraction for $F(s)$ was suggested. It has been shown recently by Ida,[7] however, that this difficulty is due to the inadequate nucleon-antinucleon pair approximation.

We may conclude this section by saying that what is relevant in deriving the Goldberger-Treiman relation is the unsubtracted dispersion relations for the form factors $f_\alpha(s)$, and we shall explicitly assume it.

Postulate II. The form factors $f_\alpha(s)$ satisfy unsubtracted dispersion relations in all channels.

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[9] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).
[10] J. Bernstein, M. Gell-Mann, and L. Michel, Nuovo Cimento **16**, 560 (1960).
[11] J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento **17**, 755 (1960).

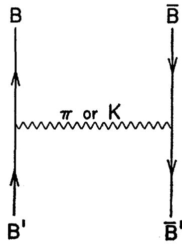

FIG. 1. The Feynman diagram expressing the final-state interaction via exchange of a π or K meson.

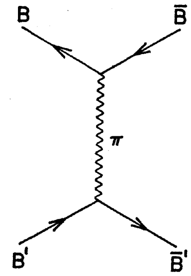

FIG. 2. The Feynman diagram expressing successive annihilation and creation of baryon-antibaryon pairs.

IV. DYNAMICAL THEORY OF FORM FACTORS

In this and following sections a dynamical calculation of the weak form factors $f_\alpha(s)$ will be carried out based on the assumed unsubtracted dispersion relations. In order to perform such a calculation, however, we have to know the T matrix for strong interactions that enters the unitarity condition. In this paper the perturbation-theoretical expression of T will be employed as an illustration, and only contributions from diagrams in Figs. 1 and 2 will be considered.

Furthermore, we shall assume that strong interactions are invariant under R conjugation defined by

$$\begin{pmatrix} p \\ n \end{pmatrix} \rightleftarrows \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix}, \quad \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \rightleftarrows \begin{pmatrix} \bar{K}^0 \\ -K^- \end{pmatrix}, \qquad (4.1)$$
$$\Sigma \to -\Sigma\,, \; \Lambda \to \Lambda\,, \; \pi \to -\pi\,.$$

This definition is different from the conventional one,[12] but as far as charge independent interactions are concerned they are essentially equivalent. This approximate invariance is not necessarily favorable in comparison with experiment, but it will be assumed for two reasons: (1) to reduce the number of independent coupling constants, and (2) to study the connection of symmetry principles between strong and weak interactions. Then, discarding the four-boson interactions the interaction Hamiltonian densities for the π- and K-couplings are given, respectively, by

$$H_\pi = iG_{NN\pi}(\bar{N}\gamma_5\boldsymbol{\tau}N - \bar{\Xi}\gamma_5\boldsymbol{\tau}\Xi)\cdot\boldsymbol{\pi} + iG_{\Sigma\Lambda\pi}(\bar{\boldsymbol{\Sigma}}\gamma_5\Lambda + \bar{\Lambda}\gamma_5\boldsymbol{\Sigma})\cdot\boldsymbol{\pi}\,, \qquad (4.2)$$

and

$$\begin{aligned} H_\pi = iG_{\Sigma NK}[&\sqrt{2}\bar{\Sigma}^+\gamma_5 p\cdot\bar{K}^0 + \bar{\Sigma}^0\gamma_5(p\cdot K^- - n\cdot\bar{K}^0) \\ &+\sqrt{2}\bar{\Sigma}^-\gamma_5 n\cdot K^- - \sqrt{2}\bar{\Sigma}^+\gamma_5\Xi^0\cdot K^+ \\ &+\bar{\Sigma}^0\gamma_5(\Xi^-\cdot K^+ + \Xi^0\cdot K^0) + \sqrt{2}\bar{\Sigma}^-\gamma_5\Xi^-\cdot K^0] \\ &+iG_{\Lambda NK}[\bar{\Lambda}^0\gamma_5(p\cdot K^- + n\cdot\bar{K}^0) \\ &\quad+\bar{\Lambda}^0\gamma_5(\Xi^-\cdot K^+ - \Xi^0\cdot K^0)] + \text{Herm. conj.} \end{aligned} \qquad (4.3)$$

In what follows we keep only the baryon-antibaryon channels and the only Feynman diagrams contributing to T are given in Figs. 1 and 2. In incorporating the requirement of postulate II into the calculation it is convenient to consider the combinations

$$D_\alpha(s) = L_\alpha(s) - FK_\alpha(s)\,. \qquad (4.4)$$

Then postulate II requires the above expressions to obey unsubtracted dispersion relations. First, the unitarity condition for $D_\alpha(s)$ will be derived.

The T matrix normalized by (2.12) is given in the lowest order approximation by

$$\begin{aligned} \langle B,B|T|B',\bar{B}'\rangle &= \langle B,\bar{B}|T^\dagger|B',\bar{B}'\rangle \\ &= A_{\alpha,\alpha'}\frac{1}{(p-p')^2+\mu^2}\bar{u}(p)i\gamma_5 u(p')\bar{v}(\bar{p}')i\gamma_5 v(\bar{p}) \\ &\quad - B_{\alpha,\alpha'}\frac{1}{(p+p')^2+\mu^2}\bar{u}(p)i\gamma_5 v(\bar{p})\bar{v}(\bar{p}')i\gamma_5 u(p')\,, \end{aligned} \qquad (4.5)$$

where A and B are constant matrices connecting various channels. The K couplings are omitted in the above formula but their contributions will be included later. The subscripts α and α' run over the following four baryon-antibaryon channels:

$$n\bar{p}, \quad \Lambda^0\bar{\Sigma}^-, \quad \Sigma^-\bar{\Lambda}^0, \quad \Xi^-\bar{\Xi}^0. \qquad (4.6)$$

The matrices A and B are given by

$$A = \begin{bmatrix} -G_{NN\pi}^2 & 0 & 0 & 0 \\ 0 & 0 & G_{\Sigma\Lambda\pi}^2 & 0 \\ 0 & G_{\Sigma\Lambda\pi}^2 & 0 & 0 \\ 0 & 0 & 0 & -G_{NN\pi}^2 \end{bmatrix}, \qquad (4.7)$$

and

$$B_{\alpha\beta} = G_\alpha G_\beta\,, \qquad (4.8)$$

respectively. The constants G_α are given by

$$G_{n\bar{p}} = \sqrt{2}G_{NN\pi}\,, \quad G_{\Lambda\bar{\Sigma}} = G_{\Sigma\bar{\Lambda}} = G_{\Sigma\Lambda\pi}\,, \quad G_{\Xi\bar{\Xi}} = -\sqrt{2}G_{NN\pi}\,. \qquad (4.9)$$

The unitarity condition for $D_\alpha(s)$ reads

$$\begin{aligned} \bar{u}(p)i\gamma_5 v(\bar{p})\,\text{Im}D_\alpha(s) = -\frac{(2\pi)^4}{2}\int\frac{d^3p'}{2p_0'}\int\frac{d^3\bar{p}'}{2\bar{p}_0'}\delta^4(p+\bar{p}-p'-\bar{p}')D_\beta(s)\Big[A_{\alpha\beta}\frac{1}{(p-p')^2+\mu^2}\bar{u}(p)i\gamma_5\frac{-ip'\gamma+M}{2p_0'} \\ \times i\gamma_5\frac{-i\bar{p}'\gamma-M}{2\bar{p}_0'}i\gamma_5 v(\bar{p}) - B_{\alpha\beta}\frac{1}{(p+p')^2+\mu^2}\bar{u}(p)i\gamma_5 v(\bar{p})\,\text{Tr}\Big(i\gamma_5\frac{-i\bar{p}'\gamma-M}{2\bar{p}_0'}i\gamma_5\frac{-ip'\gamma+M}{2\bar{p}_0'}\Big)\Big], \end{aligned} \qquad (4.10)$$

[12] S. Okubo and R. E. Marshak, Nuovo Cimento 26, 56 (1963).

where we have set all baryon masses equal to M. Since the T matrix has been evaluated in the lowest order, it is more reasonable to replace $D_\beta(s)$ by $\mathrm{Re}D_\beta(s)$ than to keep this form. Introducing a transformation of variables of integration

$$P=p+\bar{p}=p'+\bar{p}', \quad \Delta=(p-\bar{p})/2, \quad \Delta'=(p'-\bar{p}')/2, \tag{4.11}$$

we can write

$$\int\frac{d^3p'}{2p_0'}\int\frac{d^3\bar{p}'}{2\bar{p}_0'}\delta^4(p+\bar{p}-p'-\bar{p}')\cdots = \int d^4\Delta'\delta\left[\left(\frac{P}{2}+\Delta'\right)^2+M^2\right]\delta\left[\left(\frac{P}{2}-\Delta'\right)^2+M^2\right]\cdots. \tag{4.12}$$

Then making use of the formulas

$$\bar{u}(p)(ip\gamma+M)=(-i\bar{p}\gamma+M)v(\bar{p})=0, \quad p'=(P/2)+\Delta', \quad p=(P/2)+\Delta, \quad \bar{p}'=(P/2)-\Delta', \quad \bar{p}=(P/2)-\Delta, \tag{4.13}$$

we get

$$\bar{u}(p)i\gamma_5(-ip'\gamma+M)i\gamma_5(-i\bar{p}'\gamma-M)i\gamma_5v(\bar{p}) = (\Delta-\Delta')^2\bar{u}(p)i\gamma_5v(\bar{p}),$$

and

$$\mathrm{Tr}[i\gamma_5(-i\bar{p}'\gamma-M)i\gamma_5(-ip'\gamma+M)] = -2P^2=2s. \tag{4.14}$$

Inserting these results into (4.10) we get the unitarity condition. Here the final result will be written down including the contributions arising from the K-meson exchange.

$$\mathrm{Im}D_\alpha(s)=-\frac{1}{16\pi}\left(\frac{s-4M^2}{s}\right)^{1/2}\sum_\beta \mathrm{Re}D_\beta(s)\times\left[A_{\alpha\beta}\left(1-\frac{\mu^2}{s-4M^2}\ln\left(\frac{s-4M^2+\mu^2}{\mu^2}\right)\right)+C_{\alpha\beta}\left(1-\frac{\mu_K^2}{s-4M^2}\ln\left(\frac{s-4M^2+\mu_K^2}{\mu_K^2}\right)\right)+2B_{\alpha\beta}\frac{s}{s-\mu^2}\right], \tag{4.15}$$

where $C_{\alpha\beta}$ is given by

$$C=\sqrt{2}G_{\Sigma NK}G_{\Lambda NK}\begin{bmatrix}0&1&1&0\\1&0&0&-1\\1&0&0&-1\\0&-1&-1&0\end{bmatrix}. \tag{4.16}$$

In solving the equation for $D_\alpha(s)$ we have to know the boundary condition at $s=0$, which is given by

$$D_\alpha(0)=L_\alpha(0)-FK_\alpha(0)\approx f_\alpha(0)-FG_\alpha =2Mg_\alpha-FG_\alpha. \tag{4.17}$$

The constants g_α express the boundary values of the axial-vector parts.

First, let us suppose that no solution exists for the unsubtracted dispersion relations for $D_\alpha(s)$; then the only possibility is given by

$$D_\alpha(s)=0, \tag{4.18}$$

or, by the definition (4.4) we get

$$L_\alpha(s)-FK_\alpha(s)=0. \tag{4.19}$$

In terms of the field operators this relation is expressed by

$$\Phi=-\mu^2F\varphi. \tag{4.20}$$

This is exactly the model proposed by Gell-Mann and others discussed in the previous section. This does not necessarily mean, however, that the absence of the solution of the unsubtracted dispersion relations for $D_\alpha(s)$ always leads to this model, but for this model to be valid a further restrictive condition must be satisfied, i.e., the equation

$$\partial_\lambda A_\lambda=-\mu^2F\varphi \tag{4.21}$$

must be integrable to give a local axial-vector operator A_λ for the given source term. Gell-Mann and others looked for models, within the framework of Lagrangian theory, in which Eq. (4.21) is integrable. In this paper we shall look for solutions in which $D_\alpha(s)$ do not identically vanish.

Equation (4.17) involves axial-vector coupling constants g_α, and in order to discuss determination of them we shall introduce new form factors a_α and b_α by

$$\langle n\bar{p},(-)|A_\lambda(0)|0\rangle=\bar{u}(n)[a_{n\bar{p}}(s)i\gamma_\lambda\gamma_5 +b_{n\bar{p}}(s)(n+\bar{p})_\lambda\gamma_5]v(\bar{p}), \tag{4.22}$$

and similar relations for other channels. Then the form factors $f_\alpha(s)$ are expressed in terms of $a_\alpha(s)$ and $b_\alpha(s)$ by

$$f_\alpha(s)=2Ma_\alpha(s)+sb_\alpha(s), \tag{4.23}$$

and

$$g_\alpha=a_\alpha(0). \tag{4.24}$$

Therefore, determination of g_α requires the solution of the equations for $a_\alpha(s)$. The unitarity condition for $a_\alpha(s)$ can be written down when the scattering amplitudes for the baryon-antibaryon system in the ${}^3P_1$ state are known. In the present approximation we get

$$\mathrm{Im}a_\alpha(s)=-\frac{1}{32\pi}\left(\frac{s-4M^2}{s}\right)^{1/2}\sum_\beta \mathrm{Re}a_\alpha(s)\left[A_{\alpha\beta}\left(1-\frac{2\mu^2}{s-\mu^2}+\frac{1}{2}\left(\frac{2\mu^2}{s-4M^2}\right)^2\ln\left(\frac{s-4M^2+\mu^2}{\mu^2}\right)\right)+C_{\alpha\beta}\left(1-\frac{2\mu_K^2}{s-\mu_K^2}+\frac{1}{2}\left(\frac{2\mu_K^2}{s-4M^2}\right)^2\ln\left(\frac{s-4M^2+\mu_K^2}{\mu_K^2}\right)\right)\right]. \tag{4.25}$$

The vacuum polarization correction proportional to B in Eq. (4.15) does not occur in the above equation due to conservation of angular momentum in the intermediate states. In deriving the above relation, use has been made of the formulas

$$\Delta_\lambda \bar{u}(p) i\gamma_\lambda\gamma_5 v(\bar{p})=0,$$

$$\bar{u}(p)i\gamma_5(-ip'\gamma+M)i\gamma_\lambda\gamma_5(-i\bar{p}'\gamma-M)i\gamma_5 v(\bar{p}) = [(\Delta-\Delta')^2\delta_{\lambda\mu}-2(\Delta-\Delta')_\lambda(\Delta-\Delta')_\mu]\bar{u}(p)i\gamma_\mu\gamma_5 v(\bar{p}).$$

The solution of the equation for $a_\alpha(s)$ will enable us to determine the constants g_α that are needed in Eq. (4.17) provided form factors $a_\alpha(s)$ obey unsubtracted dispersion relations.

In all the existing theories, the form factors $a_\alpha(s)$ are assumed to satisfy once-subtracted dispersion relations so that the constants g_α are arbitrary, but in this paper a further assumption is made that they also satisfy unsubtracted dispersion relations in accordance with our basic ideas mentioned in Sec. I. This assumption will be formulated in a more general form by extending postulate II.

Postulate IIa. All the form factors in weak interactions satisfy unsubtracted dispersion relations.

The standard method to solve coupled integral equations of the Omnès-Muskhelishvili type is to exploit the N/D method,[13–15] but in what follows we shall solve only approximate equations by a simpler method. It is worthwhile to notice in reducing the equations that all the kernels are invariant under R conjugation and hence commute with the following matrix R:

$$R=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{4.26}$$

Furthermore, strong interactions are invariant under G conjugation so that the kernels commute with the following matrix G:

$$G=\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{4.27}$$

Presence of such quantum numbers simplifies the treatment of the problem. In Eq. (4.17) there is a term proportional to G_α which is given by

$$G_\alpha=\begin{bmatrix} \sqrt{2}G_{NN\pi} \\ G_{\Sigma\Lambda\pi} \\ G_{\Sigma\Lambda\pi} \\ -\sqrt{2}G_{NN\pi} \end{bmatrix}. \tag{4.28}$$

It is clear that this vector is a simultaneous eigenvector of R and G.

$$R(G_\alpha)=-(G_\alpha),\quad G(G_\alpha)=-(G_\alpha), \tag{4.29}$$

where the parentheses denote a vector. Therefore, we know that the vectors g_α and $D_\alpha(0)$ should have nonvanishing components, odd under both R and G. In general, both vectors are linear combinations of vectors of different transformation properties, and components of different transformation properties satisfy uncoupled sets of equations. In what follows we shall pick up only those components of g_α and $D_\alpha(0)$ that are odd under both R and G conjugations.[16] This is an additional assumption which we make in the present approximation, but if more channels, such as the three-pion channel, are introduced to improve the approximation, solutions with wrong transformation properties can be eliminated by a mechanism to be discussed later, so that this assumption becomes unnecessary. This postulate is not an essential one but it is introduced to reproduce the results of a more elaborate approximation.

V. APPROXIMATE SOLUTION

The equations for $D_\alpha(s)$ and $a_\alpha(s)$ given in the previous section are very approximate in nature, so that they serve only to illustrate the general prescription. In order to simplify the equations a further approximation is introduced by setting the masses of both the π and K mesons equal to zero; then Eqs. (4.15) and (4.25) reduce to

$$\mathrm{Im}D_\alpha(s)=-\frac{1}{32\pi}\left(\frac{s-4M^2}{s}\right)^{1/2}\sum_\beta \mathrm{Re}D_\beta(s) \times[2A_{\alpha\beta}+2C_{\alpha\beta}+4B_{\alpha\beta}], \tag{5.1}$$

and

$$\mathrm{Im}a_\alpha(s)=-\frac{1}{32\pi}\left(\frac{s-4M^2}{s}\right)^{1/2}\sum_\beta \mathrm{Re}a_\beta(s) \times[A_{\alpha\beta}+C_{\alpha\beta}], \tag{5.2}$$

respectively. First we shall find the eigenvectors of the matrices occurring in the above brackets. In accordance with the assumption made at the end of the preceding section we shall pick up only those vectors satisfying the equations

$$R(e_\alpha)=-(e_\alpha),\quad G(e_\alpha)=-(e_\alpha). \tag{5.3}$$

There are two linearly independent eigenvectors satisfying Eq. (5.3), and they are given by

$$e_1=\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},\quad e_2=\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

These two vectors span a two-dimensional vector

[13] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960).

[14] J. D. Bjorken and M. Nauenberg, Phys. Rev. **121**, 1250 (1961).

[15] S. W. MacDowell, Phys. Rev. Letters **6**, 385 (1961).

[16] M. Kawaguchi and K. Nishijima, Phys. Rev. **108**, 905 (1957). These authors considered axial-vector interactions even under R conjugation to account for the absence of the then unobserved $\pi-e$ mode of decay.

space, and projection of the matrices A, B, and C into this reduced space gives the following representations:

$$A=\begin{pmatrix} -G_1^2 & 0 \\ 0 & G_2^2 \end{pmatrix},$$

$$B=\begin{pmatrix} 4G_1^2 & 2\sqrt{2}G_1G_2 \\ 2\sqrt{2}G_1G_2 & 2G_2^2 \end{pmatrix}, \tag{5.5}$$

$$C=\begin{pmatrix} 0 & 2\sqrt{2}G_3G_4 \\ 2\sqrt{2}G_3G_4 & 0 \end{pmatrix},$$

where

$$G_1=G_{NN\pi},\quad G_2=G_{\Sigma\Lambda\pi},\quad G_3=G_{\Sigma NK},\quad G_4=G_{\Lambda NK}. \tag{5.6}$$

Therefore, in this two-dimensional reduced vector space we get

$$-A-C-2B$$
$$=\begin{pmatrix} -7G_1^2 & -2\sqrt{2}(2G_1G_2+G_3G_4) \\ -2\sqrt{2}(2G_1G_2+G_3G_4) & -5G_2^2 \end{pmatrix}, \tag{5.7}$$

and

$$-A-C=\begin{pmatrix} G_1^2 & -2\sqrt{2}G_3G_4 \\ -2\sqrt{2}G_3G_4 & -G_2^2 \end{pmatrix}. \tag{5.8}$$

The eigenvectors of these matrices should satisfy decoupled Omnès equations, and the absence of subtractions requires that only those eigenvectors belonging to positive eigenvalues are the accessible solutions. Call the positive eigenvalues of (5.7) and (5.8), λ_1 and λ_3, respectively; then the s dependence of the solutions is given by

$$D_\alpha(s)\propto\exp\left[\frac{s}{\pi}\int_{4M^2}^{\infty}\frac{ds'}{s'(s'-s-i\epsilon)}\times\tan^{-1}\left(\frac{\lambda_1}{16\pi}\left(\frac{s'-4M^2}{s'}\right)^{1/2}\right)\right], \tag{5.9}$$

$$a_\alpha(s)\propto\exp\left[\frac{s}{\pi}\int_{4M^2}^{\infty}\frac{ds'}{s'(s'-s-i\epsilon)}\times\tan^{-1}\left(\frac{\lambda_3}{32\pi}\left(\frac{s'-4M^2}{s'}\right)^{1/2}\right)\right], \tag{5.10}$$

respectively. The subscripts 1 and 3 to λ stand for the ${}^1S_0$ and ${}^3P_1$ of the baryon-antibaryon system. The positive eigenvalues and the corresponding eigenvectors are given by

$$\lambda_1=\tfrac{1}{2}[-7G_1^2-5G_2^2+((7G_1^2-5G_2^2)^2+32(2G_1G_2+G_3G_4)^2)^{1/2}],$$

$$\lambda_3=\tfrac{1}{2}[G_1^2-G_2^2+((G_1^2+G_2^2)^2+32G_3^2G_4^2)^{1/2}],$$

$$D_\alpha(0)=(2Mg_A-\sqrt{2}FG_1)(e_1+c_1e_2)_\alpha,$$

with

$$c_1=-\frac{7G_1^2-5G_2^2+[(7G_1^2-5G_2^2)^2+32(2G_1G_2+G_3G_4)^2]^{1/2}}{4\sqrt{2}(2G_1G_2+G_3G_4)},$$

and

$$g_\alpha=a_\alpha(0)=g_A(e_1+c_3e_2)_\alpha,$$

with

$$c_3=\frac{G_1^2+G_2^2-[(G_1^2+G_2^2)+32G_3^2G_4]^{1/2}}{4\sqrt{2}G_3G_4}. \tag{5.11}$$

It should be mentioned, however, that the eigenvalue λ_1 is positive only when

$$\left|\frac{G_3G_4}{G_1G_2}+2\right|>\left(\frac{35}{8}\right)^{1/2}. \tag{5.12}$$

This restriction arises from the fact that the vacuum-polarization contribution in (5.7) expressed by the matrix B gives a strong repulsion and in order to overcome this repulsion the nondiagonal matrix elements have to play an important role in providing a strong attraction. In fact, in the nucleon-antinucleon approximation considered by Federbush, Goldberger, and Treiman,[17] there is no positive eigenvalue.

[17] P. Federbush, M. L. Goldberger, and S. B. Treiman, Phys. Rev. **112**, 642 (1958), see Appendix B.

Next, the above results will be combined with

$$D_\alpha(0)\approx 2Mg_\alpha-FG_\alpha \tag{4.17}$$

and

$$G_\alpha=(G_1e_1+G_2e_2)_\alpha. \tag{5.13}$$

Since three two-dimensional vectors $D_\alpha(0)$, g_α, and G_α are linearly dependent, we can determine the constant F in terms of other parameters.

$$2Mg_A-\sqrt{2}FG_1=\frac{G_2-\sqrt{2}G_1c_3}{c_3-c_1}F. \tag{5.14}$$

It is extremely important that due to R-conjugation invariance all three vectors above become essentially two dimensional and only one constant F is determined. Should the R invariance be invalid, we would in general get more restrictions on the choice of parameters. This is one of the reasons we dropped solutions even under R conjugation. It is clear now that the so-called Goldberger-Treiman relation holds when and only when the right-hand side of Eq. (5.14) is very small as compared with $\sqrt{2}FG_1$.

What is remarkable here is the fact that we can get another equation to determine F in terms of other parameters with the help of Ida's formula (3.15). The results obtained in this section are based essentially on the convergence condition that is independent of Ida's formula.

Finally, it should be mentioned that unsubtracted dispersion relations are more easily satisfied by the axial-vector form factors $a_\alpha(s)$ than by the pseudoscalar from factors $D_\alpha(s)$ as seen from the presence of a restriction (5.12) for the latter, which is a justification of postulate IIa introduced in this paper.

VI. APPLICATION OF IDA'S FORMULA

In order to apply the unsubtracted dispersion relation for $F(s)$ to the solution obtained in the preceding section we shall exploit Ida's formula (3.15). For this purpose the following expressions will be evaluated first:

$$\sigma(s)=\frac{1}{8\pi^2}\frac{[s(s-4M^2)]^{1/2}}{(s-\mu^2)^2}\sum_\alpha |K_\alpha(s)|^2, \tag{6.1}$$

and

$$\gamma(s)-F\sigma(s)=\frac{1}{8\pi^2}\frac{[s(s-4M^2)]^{1/2}}{(s-\mu^2)^2}\sum_\alpha K_\alpha^*(s)D_\alpha(s). \tag{6.2}$$

The vector $K_\alpha(s)$ has two components, one parallel to $D_\alpha(s)$ and the other orthogonal to $D_\alpha(s)$. The component orthogonal to $D_\alpha(s)$ is an increasing function of s, and the parallel component being proportional to $D_\alpha(s)$ is a decreasing function of s. In the expression (6.2) only the decreasing component of $K_\alpha(s)$ contributes to the scalar product with $D_\alpha(s)$. Inserting the solution for $D_\alpha(s)$ into (6.2), we get

$$\begin{aligned}\gamma(s)-F\sigma(s)&=\frac{1}{4\pi^2}\frac{[s(s-4M^2)]^{1/2}}{(s-\mu^2)^2}(2Mg_A-\sqrt{2}FG_1)(\sqrt{2}G_1+c_1G_2)\\&\times\exp\left[\frac{2s}{\pi}\int\frac{ds'}{s'(s'-s)}\times\tan^{-1}\left(\frac{\lambda_1}{16}\left(\frac{s'-4M^2}{s'}\right)^{1/2}\right)\right].\end{aligned} \tag{6.3}$$

In evaluating the denominator in Ida's formula we shall use perturbation theory to show that the second term in the denominator is small as compared with unity.

$$\begin{aligned}1+\int_{4M^2}^\infty\frac{\mu^2}{s}\sigma(s)ds&\approx 1+\int_{4M^2}^\infty ds\frac{\mu^2}{s}\frac{1}{8\pi^2}\frac{[s(s-4M^2)]^{1/2}}{(s-\mu^2)^2}\sum_\alpha G_\alpha^2\\&\approx 1+\frac{1}{6\pi}\left(\frac{\mu}{M}\right)^2\sum_\alpha\frac{G_\alpha^2}{4\pi}\\&\approx 1.\end{aligned} \tag{6.4}$$

Thus the Ida formula is given by

$$\begin{aligned}F\approx(\sqrt{2}G_1+c_1G_2)\frac{1}{4\pi^2}\int_{4M^2}^\infty ds\frac{1}{s}\left(\frac{s-4M^2}{s}\right)^{1/2}\\\times\exp\left[\frac{2s}{\pi}\int_{4M^2}^\infty\frac{ds'}{s'(s'-s)}\tan^{-1}\left(\frac{\lambda_1}{16\pi}\left(\frac{s'-4M^2}{s'}\right)^{1/2}\right)\right]\\\times(2Mg_A-\sqrt{2}FG_1).\end{aligned} \tag{6.5}$$

Since the Goldberger-Treiman relation requires

$$\lambda_1/16\pi\ll 1 \tag{6.6}$$

as we shall see soon, it will be assumed to further approximate (6.5). With the assumption (6.6), the integral in (6.5) is simplified and is given by

$$\frac{1}{4\pi^2}\int_{4M^2}^\infty\frac{ds}{s}\left(\frac{s-4M^2}{s}\right)^{1/2}\exp\left[\frac{2s}{\pi}\int_{4M^2}^\infty\frac{ds'}{s'(s'-s)}\times\tan^{-1}\left(\frac{\lambda_1}{16\pi}\left(\frac{s'-4M^2}{s'}\right)^{1/2}\right)\right]\approx\frac{2}{\lambda_1}. \tag{6.7}$$

Then Eq. (6.5) reduces to

$$2Mg_A-\sqrt{2}FG_1=\frac{\lambda_1}{2(\sqrt{2}G_1+c_1G_2)}F. \tag{6.8}$$

The Goldberger-Treiman relation is obtained from (6.8) by dropping the right-hand side, which is consistent with (6.6). It is interesting to compare (6.8) with (5.14) since combination of them gives

$$\frac{\lambda_1}{2(\sqrt{2}G_1+c_1G_2)}=\frac{G_2-\sqrt{2}c_3G_1}{c_3-c_1}. \tag{6.9}$$

All the parameters involved in this equation are expressible in terms of strong coupling constants, so that Eq. (6.9) expresses an eigenvalue restriction imposed on the choice of the fundamental parameters in strong interactions. Remarkable results here are that the weak interactions are completely determined, except for the over-all normalization, by strong interactions and that the presence of weak interactions alone imposes eigenvalue restrictions on the strong-interaction parameters. In other words, strong and weak interactions are controlled by each other and their structure cannot be discussed separately.

It is a very interesting problem to solve the eigenvalue equation to determine one coupling constant assuming others, but we shall not try to do it in the present paper since the solution of the eigenvalue problem is extremely sensitive to the approximation employed. First, in deriving the eigenvalue equation, low-energy behavior of the solution of the integral equation is important, but the approximation employed in Sec. V is good only at high energies since those terms in Eqs. (4.15) and (4.25) neglected in writing down Eqs. (5.1) and (5.2) give important contributions at low energies. Second, the contributions from vector mesons have been completely neglected in this paper. If the vector mesons should be responsible, at least partly, for the hard core in nuclear forces, their contributions at high energies should not be overlooked. In order to illustrate the effects of vector mesons on the present problem, let us consider the contributions of the ω meson. Assuming that the ω meson is odd under R conjugation, we shall write down the following interaction of the ω meson with baryons:

$$H_\omega = iG_{NN\omega}(\bar{N}\gamma_\lambda N - \bar{\Xi}\gamma_\lambda\Xi)\omega_\lambda, \tag{6.10}$$

then this interaction modifies the bracket in Eq. (4.15) by an additional term

$$-2G_{NN\omega}^2\begin{bmatrix}1 & & \\ & 0 & \\ & & 0 & \\ & & & 1\end{bmatrix}\frac{s-2M^2}{s-4M^2}\ln\left(\frac{s-4M^2+\mu_\omega^2}{\mu_\omega^2}\right), \tag{6.11}$$

and it also changes the bracket in Eq. (4.25) by

$$-4G_{NN\omega}^2\begin{bmatrix}1 & & \\ & 0 & \\ & & 0 & \\ & & & 1\end{bmatrix}\frac{1}{s-4M^2}$$

$$\times\left[-s+\left(s-2M^2+\mu_\omega^2\frac{s}{s-4M^2}\right)\right.$$

$$\left.\times\ln\left(\frac{s-4M^2+\mu_\omega^2}{\mu_\omega^2}\right)\right]. \tag{6.12}$$

These terms dominate other terms in the brackets at high energies, at least, in the present approximation. Of course, one should be reminded of the possible damping of these terms at high energies due to vertex corrections. At any rate it is extremely hard to derive a reliable eigenvalue equation which we can work with.

VII. NATURE OF THE EIGENVALUE PROBLEM

One of the most important achievements attained in this paper is the derivation of the eigenvalue restriction. It has been known for some time,[18] however, that the assumption of unsubtracted dispersion relations alone is not sufficient to get this restriction so that one has to find a reason, case by case, why the eigenvalue restriction is obtained. Therefore, we shall recapitulate the arguments leading to the eigenvalue restriction in the two already discussed cases of nonleptonic decays of hyperons and of $\pi-\mu$ decay.

In the former case we dealt with vertex functions corresponding to

$$Y\to N+\pi. \tag{7.1}$$

There are different ways, however, to define form factors according to the choice of the particle to be put off the mass shell. In the integral equations different types of form factors are coupled through unitarity and as far as the integral equations are concerned these vertex functions appear as completely independent objects. The eigenvalue restriction arises from the fact that different types of form factors must be equal when all the three particles are on the mass shell.

In the $\pi-\mu$ decay problem the eigenvalue restriction arises for a different reason. An important point is that the axial vector operator A_λ is not irreducible but has a pseudoscalar component $\partial_\lambda A_\lambda$ so that we get Eq. (4.17). The ratio of the components of $D_\alpha(0)$ is determined by the strong final-state interaction of the baryon-antibaryon system in the $^1S_0$ state, whereas that of g_α is determined by the interaction in the $^3P_1$ state. Thus Eq. (4.17) renders determination of the constant F in terms of other parameters, and the assumed unsubtracted dispersion relation for $F(s)$ gives another expression of F in terms of the other parameters. Then a consistency requirement is given, in the form of an eigenvalue equation, that the two expressions for F be identical.

There are a few further remarks deserving emphasis. The restriction brought about by Eq. (4.17) is really a very strong one, but in the approximation employed in this paper we could easily adjust the parameters so as to satisfy Eq. (4.17). This is because both $D_\alpha(0)$ and g_α can be formed by superposing two vectors e_1 and e_2, but in general more independent vectors might be needed to express $D_\alpha(0)$, g_α, and G_α, if the R invariance were not valid. This means that the restriction of the theory by Eq. (4.17) is so strong that more than one constant can be determined after all. The restriction other than the eigenvalue condition (6.9) did not appear explicitly in the present approximation since the other condition is automatically satisfied by the assumed R invariance for strong interactions. This suggests that the existence of the solution of the problem of weak interactions requires some symmetry higher than charge independence in one form or the other. From the discussion above, it is clear that the situation is completely different for conserved vector and pseudoscalar couplings. For the former, the scalar part is absent so that the final-state interactions are relevant only in the $J=1$ states, while for the latter only $J=0$ is relevant. In either case only one value of the angular momentum enters the final-state interactions.

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[18] M. Baker and F. Zachariasen, Phys. Rev. **119**, 438 (1960).

Pseudoscalar Coupling

It has been pointed out already by Ida that an unsubtracted dispersion relation for $F(s)$ is unlikely for the pseudoscalar Fermi interaction. Because of the importance of this question his argument will be repeated here.

Let us start from the following effective Hamiltonian or the S matrix:

$$H_P = i\Phi'\bar{\psi}_\nu(1-\gamma_5)\psi_\mu + \text{Herm. conj.} \tag{7.2}$$

Then introduce

$$\langle\pi^-|\Phi'(0)|0\rangle = -\frac{1}{(2q_0)^{1/2}}m_\mu F \equiv -\frac{1}{(2q_0)^{1/2}}\mu^2 F', \tag{7.3}$$

$$\langle\alpha(-)|\Phi'(0)|0\rangle = c_\alpha f_\alpha'(s), \tag{7.4}$$

and other formulas corresponding to the axial-vector case, in particular, the formula

$$\langle''\pi^-''(-)|\Phi'(0)|0\rangle = -\frac{\mu^2}{(2q_0)^{1/2}}F'(s). \tag{7.5}$$

Ida's formula in this case is given by

$$F' = \frac{\int ds[(s-\mu^2)/\mu^2][\gamma'(s) - F'\sigma(s)]}{1+\int ds\sigma(s)}. \tag{7.6}$$

In this formula the second term in the numerator is as divergent as the pion self-energy and it is very unlikely that this divergence is cancelled by the first term. Therefore, it seems reasonable to assume that the function $F'(s)$ requires at least one subtraction. Furthermore, we do not get any eigenvalue restriction in the same approximation as we employed in the axial-vector case. The presence of the pseudoscalar coupling thus contradicts our basic assumption that all the weak amplitudes are governed by unsubtracted dispersion relations.

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